

# Permutations of the integers induce only the trivial automorphism of the Turing degrees

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## Abstract

Let  $\pi$  be an automorphism of the Turing degrees induced by a homeomorphism  $\varphi$  of the Cantor space  $2^\omega$  such that  $\varphi$  preserves all Bernoulli measures. It is proved that  $\pi$  must be trivial. In particular, a permutation of  $\omega$  can only induce the trivial automorphism of the Turing degrees.

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# 1 Introduction

Let  $\mathcal{D}_T$  denote the set of Turing degrees and let  $\leq$  denote its ordering. This article gives a partial answer to the following famous question.

*Question 1.* Does there exist a nontrivial automorphism of  $\mathcal{D}_T$ ?

**Definition 1.** A bijection  $\pi : \mathcal{D}_T \rightarrow \mathcal{D}_T$  is an *automorphism* of  $\mathcal{D}_T$  if for all  $\mathbf{x}, \mathbf{y} \in \mathcal{D}_T$ ,  $\mathbf{x} \leq \mathbf{y}$  iff  $\pi(\mathbf{x}) \leq \pi(\mathbf{y})$ . If moreover there exists an  $\mathbf{x}$  with  $\pi(\mathbf{x}) \neq \mathbf{x}$  then  $\pi$  is *nontrivial*.

Question 1 has a long history. Already in 1977, Jockusch and Solovay [3] showed that each jump-preserving automorphism of the Turing degrees is the identity above  $\mathbf{0}^{(4)}$ . Nerode and Shore 1980 [8] showed that each automorphism (not necessarily jump-preserving) is equal to the identity on some cone. Slaman and Woodin [11] showed that each automorphism is equal to the identity on the cone above  $\mathbf{0}''$ .

Haught and Slaman [2] used permutations of the integers to obtain automorphisms of the polynomial-time Turing degrees in an ideal (below a fixed set).

**Theorem 2** (Haught and Slaman [2]). *There is a permutation of  $2^{<\omega}$ , or equivalently of  $\omega$ , that induces a nontrivial automorphism of*

$$(\text{PTIME}^A, \leq_{\text{PT}}).$$

for some  $A$ .

Our result can be seen as a contrast to the following work of Kent.

**Definition 3.**  $A \subset \omega$  is *cohesive* if for each recursively enumerable set  $W_e$ , either  $A \cap W_e$  is finite or  $A \cap (\omega \setminus W_e)$  is finite.

**Theorem 4** (Kent [9, Theorem 12.3.IX], [4, 5]). *There exists a permutation  $f$  such that*

- (i) *for all recursively enumerable  $B$ ,  $f(B)$  and  $f^{-1}(B)$  are recursively enumerable (and hence for all recursive  $A$ ,  $f(A)$  and  $f^{-1}(A)$  are recursive);*
- (ii)  *$f$  is not recursive.*

*Proof.* Kent's permutation is just any permutation of a cohesive set (and the identity off the cohesive set).  $\square$

## 2 Universal algebra setup

**Definition 5.** The *pullback* of  $f : \omega \rightarrow \omega$  is  $f^* : \omega^\omega \rightarrow \omega^\omega$  given by

$$f^*(A)(n) = A(f(n)).$$

We often write  $F = f^*$ . Given a set  $S \subseteq \omega$  let  $\mathcal{D}_S = S^\omega / \equiv_T$ . Thus the elements of  $\mathcal{D}_S$  are of the form

$$[g]_S = \{ h \in S^\omega \mid h \equiv_T g \}, \quad g \in S^\omega.$$

Given  $F : S^\omega \rightarrow S^\omega$ , let  $F_S : \mathcal{D}_S \rightarrow \mathcal{D}_S$  be defined by

$$F_S([A]_S) = [F(A)]_S.$$

If  $F = f_S^*$  then we say that  $F_S$  and  $F$  are both *induced* by  $f$ .

**Lemma 6.** *For each  $f : \omega \rightarrow \omega$  and each  $S \subseteq \omega$ , the pullback  $f^*$  maps  $S^\omega$  into  $S^\omega$ .*

*Proof.*

$$A \in S^\omega, n \in \omega \implies f^*(A)(n) = A(f(n)) \in S. \quad \square$$

In light of Lemma 6, we can define:

**Definition 7.**  $f_S^* : \mathcal{D}_S \rightarrow \mathcal{D}_S$  is the map given by

$$f_S^*([g]_S) = [f^*(g)]_S.$$

For  $S \subseteq \omega$  (with particular attention to  $S \in \{2, \omega\}$ ), let

$$\mathcal{D}_S = S^\omega / \equiv_T.$$

Our main result concerns  $\mathcal{D}_2$ ; the corresponding result for  $\mathcal{D}_\omega$  is much easier:

**Theorem 8.** *Let  $f : \omega \rightarrow \omega$  be a bijection and let  $f^*$  be its pullback. If  $f_S^*$  is an automorphism of  $\mathcal{D}_S$  for some infinite computable set  $S$ , then  $f$  is computable.*

*Proof.* Let  $\eta : \omega \rightarrow S$  be a computable bijection between  $\omega$  and  $S$ . Then for all  $x \in \omega$ ,

$$f^*(\eta \circ f^{-1})(x) = (\eta \circ f^{-1})(f(x)) = \eta(f^{-1}(f(x))) = \eta(x).$$

Since  $\eta \in S^\omega$  is computable and  $f_S^*$  is an automorphism,  $\eta \circ f^{-1} \in S^\omega$  must be computable. Hence  $f$  is computable.  $\square$

### 3 Permutations preserve randomness

**Theorem 9.** *If  $B$  is  $f$ - $\mu_p$ -random,  $F = f^*$  and  $A = F(B)$  or  $A = F^{-1}(B)$ , then  $A$  is  $f$ - $\mu_p$ -random.*

*Proof.* First note that  $f^{-1}$ - $\mu_p$ -randomness is the same as  $f$ - $\mu_p$ -randomness since  $f \equiv_T f^{-1}$ . Thus the result for  $A = F^{-1}(B)$  follows from the result for  $A = F(B)$ . So suppose  $A = F(B)$  and  $A$  is not  $f$ - $\mu_p$ -random. So  $A \in \cap_n U_n$  where  $\{U_n\}_n$  is an  $f$ - $\mu_p$ -ML test. Then

$$B \in \{X \mid F(X) \in \cap_n U_n\} = \cap_n V_n$$

where

$$V_n = \{X \mid F(X) \in U_n\} = F^{-1}(U_n)$$

We claim that  $V_n$  is  $\Sigma_1^0(f)$  (uniformly in  $n$ ) and  $\mu_p(V_n) = \mu_p(U_n)$ . Write  $U_n = \cup_k [\sigma_k]$  where the strings  $\sigma_k$  are all incomparable. Then

$$V_n = \cup_k F^{-1}([\sigma_k])$$

and

$$\mu_p[\sigma_k] = \mu_p F^{-1}([\sigma_k])$$

and the  $F^{-1}([\sigma_k])$ ,  $k \in \omega$  are still disjoint and clopen. (If we think of  $\sigma \in 2^{<\omega}$  as a partial function from  $\omega$  to 2 then

$$\begin{aligned} F^{-1}([\sigma]) &= \{X \mid F(X) \in [\sigma]\} \\ &= \{X \mid X(f(n)) = \sigma(n), n < |\sigma|\} = [\{\langle f(n), \sigma(n) \rangle \mid n < |\sigma|\}] \end{aligned}$$

Thus  $\{V_n\}_n$  is another  $f$ - $\mu_p$ -ML test, and so  $B$  is not  $f$ - $\mu_p$ -random, which completes the proof.  $\square$

**Theorem 10.**  $\mu_p(\{A : A \geq_T p\}) = 1$ , in fact if  $A$  is  $\mu_p$ -ML-random then  $A$  computes  $p$ .

*Proof.* Kjos-Hanssen [6] showed that each Hippocratic  $\mu_p$ -random set computes  $p$ . In particular, each  $\mu_p$ -random set computes  $p$ .  $\square$

## 4 Cones have small measure

**Definition 11** (Bernoulli measures). For each  $n \in \omega$ ,

$$\mu_p(\{X \in 2^\omega : X(n) = 1\}) = p$$

and  $X(0), X(1), X(2), \dots$  are mutually independent random variables.

**Definition 12.** An *ultrametric* space is a metric space with metric  $d$  satisfying the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

**Definition 13.** A *Polish space* is a separable completely metrizable topological space.

**Definition 14.** In a metric space,  $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ .

**Theorem 15** ([7, Proposition 2.10]). Suppose that  $X$  is a Polish ultrametric space,  $\mu$  is a probability measure on  $X$ , and  $\mathcal{A} \subseteq X$  is Borel. Then

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu(\mathcal{A} \cap B(x, \varepsilon))}{\mu(B(x, \varepsilon))} = 1$$

for  $\mu$ -almost every  $x \in \mathcal{A}$ .

**Definition 16.** For any measure  $\mu$  define the conditional measure by

$$\mu(\mathcal{A} \mid \mathcal{B}) = \frac{\mu(\mathcal{A} \cap \mathcal{B})}{\mu(\mathcal{B})}.$$

A measurable set  $\mathcal{A}$  has density  $d$  at  $X$  if

$$\lim_n \mu_p(\mathcal{A} \mid [X \upharpoonright n]) = d.$$

Let  $\Xi(\mathcal{A}) = \{X : \mathcal{A} \text{ has density 1 at } X\}$ .

**Theorem 17** (Lebesgue Density Theorem for  $\mu_p$ ). *For Cantor space with Bernoulli( $p$ ) product measure  $\mu_p$ , the Lebesgue Density Theorem holds:*

$$\lim_{n \rightarrow \infty} \frac{\mu_p(\mathcal{A} \cap [x \upharpoonright n])}{\mu_p([x \upharpoonright n])} = 1$$

for  $\mu$ -almost every  $x \in \mathcal{A}$ .

If  $\mathcal{A}$  is measurable then so is  $\Xi(\mathcal{A})$ . Furthermore, the measure of the symmetric difference of  $\mathcal{A}$  and  $\Xi(\mathcal{A})$  is zero, so  $\mu(\Xi(\mathcal{A})) = \mu(\mathcal{A})$ .

*Proof.* Consider the ultrametric  $d(x, y) = 2^{-\min\{n: x(n) \neq y(n)\}}$ . It induces the standard topology on  $2^\omega$ . Apply Theorem 15.  $\square$

Sacks [10] and de Leeuw, Moore, Shannon, and Shapiro [1] showed that each cone in the Turing degrees has measure zero. Here we use Theorem 17 to extend this to  $\mu_p$ .

**Theorem 18.** *If  $\mu_p(\{X : W_e^X = A\}) > 0$  then  $A$  is c.e. in  $p$ .*

*Proof.* Suppose  $\mu_p(\{X : W_e^X = A\}) > 0$ . Then  $S := \{X \mid W_e^X = A\}$  has positive measure, so  $\Xi(S)$  has positive measure, and hence by Theorem 15 there is an  $X$  such that  $S$  has density 1 at  $X$ . Thus, there is an  $n$  such that  $\mu_p(S \mid [X \upharpoonright n]) > \frac{1}{2}$ . Let  $\sigma = X \upharpoonright n$ . We can now enumerate  $A$  using  $p$  by taking a “vote” among the sets extending  $\sigma$ . More precisely,  $n \in A$  iff

$$\mu_p(\{Y : \sigma \prec Y \wedge n \in W_e^Y\}) > \frac{1}{2},$$

and the set of  $n$  for which this holds is clearly c.e. in  $p$ .  $\square$

**Theorem 19.** *Each cone strictly above  $p$  has  $\mu_p$ -measure zero:*

$$\mu_p(\{A : A \geq_T q\}) = 1 \quad \implies \quad q \leq_T p.$$

*Proof.* If  $A$  can compute  $q$  then  $A$  can enumerate both  $q$  and the complement of  $q$ . Hence by Theorem 18,  $q$  is both c.e. in  $p$  and co-c.e. in  $p$ ; hence  $q \leq_T p$ .  $\square$

## 5 Main result

We are now ready to prove our main result Theorem 20 that no nontrivial automorphism of the Turing degrees is induced by a permutation of  $\omega$ .

**Theorem 20.** *If  $\pi$  is an automorphism of  $\mathcal{D}_2$  which is induced by a permutation of  $\omega$  then  $\pi(\mathbf{p}) = \mathbf{p}$  for each  $\mathbf{p} \in \mathcal{D}_T$ .*

*Proof.* Fix a permutation  $f : \omega \rightarrow \omega$  and let  $F = f^* \upharpoonright 2^\omega$ . Let  $B$  be  $f$ - $\mu_p$ -random. We claim that  $B$  computes  $F(p)$ .

By Theorem 10, for any  $f$ - $\mu_p$  random  $A$ , we have  $p \leq_T A$ , hence  $F(p) \leq_T F(A)$ . So it suffices to represent  $B$  as  $F(A)$ .

Now  $B = F(F^{-1}(B))$ . Let  $A = F^{-1}(B)$ . By Theorem 9,  $A$  is  $f$ - $\mu_p$ -random. Thus every  $f$ - $\mu_p$ -random computes  $F(p)$ .

Thus we have completed the proof of our claim that  $\mu_p$ -almost every real computes  $F(p)$ .

By Theorem 19 it follows that  $F(p) \leq_T p$ .

By considering the inverse  $f^{-1}$  we also obtain  $F^{-1}(p) \leq_T p$  and hence  $p \leq_T F(p)$ . So  $F(p) \equiv_T p$  and  $F$  induces the identity automorphism.  $\square$

## 6 Computing the permutation

**Theorem 21.** *Let  $f : \omega \rightarrow \omega$  be a permutation. Let  $F = f^*$  be its pullback (Definition 5) to  $2^\omega$ . If for positive Lebesgue measure many  $G$ ,  $F(G) \leq_T G$ , then  $f$  is recursive.*

*Proof.* By the Lebesgue Density Theorem we can get a  $\Phi$  and a  $\sigma$  such that, if  $\mu_\sigma$  denotes conditional probability on  $\sigma$  and  $E = \{A : F(A) = \Phi^A\}$ , then

$$\mu_\sigma(E) \geq 95\%.$$

For simplicity let us write  $p_n(A) = A+n = A \cup \{n\}$  and  $m_n(A) = A-n = A \setminus \{n\}$ . Then  $p_n^{-1}E = \{A : p_n(A) \in E\}$ . Note that

$$E \subseteq p_n^{-1}(E) \cup m_n^{-1}(E)$$

and

$$E^c \subseteq p_n^{-1}(E^c) \cup m_n^{-1}(E^c)$$

Then

$$\mu_\sigma(E) \leq \mu_\sigma(p_n^{-1}(E) \cup m_n^{-1}(E)) \leq \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E))$$

We now have

$$\mu_\sigma\{A : F(A+n) = \Phi^{A+n}\} \geq 90\%$$

and

$$\mu_\sigma\{A : F(A-n) = \Phi^{A-n}\} \geq 90\%;$$

Indeed, the events  $m_n^{-1}(A)$ ,  $p_n^{-1}(A)$  are each independent of the event  $n \in A$ , so for  $n > |\sigma|$ ,

$$\begin{aligned} 95\% \leq \mu_\sigma(E) &= \mu_\sigma(p_n^{-1}(E) \mid n \in A) \mu_\sigma(n \in A) + \mu_\sigma(p_n^{-1}(E) \mid n \notin A) \mu_\sigma(n \notin A) \\ &= \frac{1}{2} (\mu_\sigma(p_n^{-1}(E) \mid n \in A) + \mu_\sigma(m_n^{-1}(E) \mid n \notin A)) \\ &= \frac{1}{2} (\mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E))) \end{aligned}$$

which gives

$$1.9 \leq \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E)) \leq 1 + \min\{\mu_\sigma(p_n^{-1}(E)), \mu_\sigma(m_n^{-1}(E))\}.$$

Also  $F(A - n)$  and  $F(A + n)$  differ in exactly one bit, namely  $f^{-1}(n)$ , for all  $A$ :

$$\begin{aligned} F(A - n)(b) \neq F(A + n)(b) &\iff (A - n)(f(b)) \neq (A + n)(f(b)) \\ &\iff n = f(b) \iff b = f^{-1}(n), \end{aligned}$$

that is

$$\{A : (\forall b)(F(A + n)(b) \neq F(A - n)(b) \leftrightarrow b = f^{-1}(n))\} = 2^\omega.$$

Let  $D_{n,b} = \{A : \Phi^{A+n}(b) \downarrow \neq \Phi^{A-n}(b) \downarrow\}$ . For  $n > |\sigma|$ ,

$$\mu_\sigma \left( D_{n,f^{-1}(n)} \setminus \bigcup_{b \neq f^{-1}(n)} D_{n,b} \right) = \mu_\sigma \{A : (\forall b)(A \in D_{n,b} \leftrightarrow b = f^{-1}(n))\} \geq 80\%$$

since

$$\begin{aligned} &\mu_\sigma \{A : \neg(\forall b)(A \in D_{n,b} \leftrightarrow b = f^{-1}(n))\} \\ &\leq \mu_\sigma(\neg p_n^{-1}(E)) + \mu_\sigma(\neg m_n^{-1}(E)) \leq 10\% + 10\% = 20\%. \end{aligned}$$

Therefore, given any  $n$ , we can compute  $f^{-1}(n)$ : enumerate computations until we have found some bit  $b$  such that

$$\mu_\sigma D_{n,b} \geq 80\%.$$

Then  $b = f^{-1}(n)$ .

Thus  $f^{-1}$  is computable and hence so is  $f$ . □

**Theorem 22.** *If  $\pi$  is an automorphism of  $\mathcal{D}_T$  which is induced by a permutation  $f$  of  $\omega$  then  $f$  is recursive.*

*Proof.* By Theorem 20,  $f^*(G) \equiv_T G$  for each  $G \in 2^\omega$ . By Theorem 21,  $f$  is recursive. □

## 7 Measure-preserving homeomorphisms of the Cantor set

**Proposition 23.** *A permutation of  $\omega$  induces a homeomorphism of  $2^\omega$  that is  $\mu_p$ -preserving for each  $p$ .*

**Proposition 24.** *There exist homeomorphisms of  $2^\omega$  that are  $\mu_p$ -preserving for each  $p$ , but are not induced by a permutation.*

*Proof.* Map

$$[1] \mapsto [111] \cup [001] \cup [101] \cup [110]$$

(more generally, any collection of cylinders of strings of length 3 including 2 strings of Hamming weight 2 and 1 of Hamming weight 1).

Another way to express this is that the homeomorphism preserves the fraction of 1s in a certain sense.

More precisely,

$$\begin{aligned} 100 &\mapsto 001, \\ 101 &\mapsto 101, \\ 110 &\mapsto 110, \\ 111 &\mapsto 111. \end{aligned}$$

□

**Theorem 25.** *Suppose  $\varphi$  is a homeomorphism of  $2^\omega$  which is  $\mu_p$ -preserving for all  $p$  (it suffices to require this for infinitely many  $p$ , or for a single transcendental  $p$ ). Suppose  $\varphi$  induces an automorphism  $\pi$  of the Turing degrees. Then  $\pi = \text{id}$ .*

We omit the proof which follows along the same lines as before.



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